2.3b local stability of first order equations

Monday, January 25, 2021 2:12 AM

Lemma (Translation): Let $\bar{x}$ be an equilibrium of $x_{t+1} = f(x_t)$.
Define a variable $u_t = x_t - \bar{x}$. Then $\bar{u} = 0$ is an equilibrium of $u_{t+1} = g(u_t)$, where $g(u) = f(u + \bar{x}) - f(\bar{x})$. Furthermore, 0 is locally stable (or unstable, or locally asymptotically stable) fixed pt of g iff $\bar{x}$ " " " " " " " " " " " " " f.

$$u_{t+1} = x_{t+1} - \bar{x} = f(x_t) - \bar{x} = f(u_t + \bar{x}) - f(\bar{x}) = g(u_t)$$

Consider: Suppose $\exists f''$ continuous on an open interval $I \ni \bar{x}$. Then by Taylor's Thm,

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + \frac{f''(\xi)}{2!}(x - \bar{x})^2$$

for some $\xi \in I$. If $(x_t - \bar{x})$ is small, can approximate

$$\left.\begin{aligned} f(x_t) - \bar{x} &\approx f'(\bar{x})(x - \bar{x}) \\ \text{or} \quad u_{t+1} &\approx f'(\bar{x})\, u_t \end{aligned}\right\} \text{linear approximation at } \bar{x}$$

We don't actually generally need f''.

Thm 2.1 Let f have a continuous first derivative f' on an open interval $I \ni \bar{x}$, and $\bar{x}$ is a fixed pt of f.
Then $\bar{x}$ is a locally asymptotically stable equilibrium of $x_{t+1} = f(x_t)$ if

$$|f'(\bar{x})| < 1$$

and unstable if $|f'(\bar{x})| > 1$.

proof. Case 1: $|f'(\bar{x})| < 1$. Because f' is continuous on I, can choose

$$[\bar{x} - \varepsilon, \bar{x} + \varepsilon] \subset I \text{ s.t. } |f'(x)| < c < 1 \text{ for } x \in [\bar{x} - \varepsilon, \bar{x} + \varepsilon].$$

By the Mean Value Thm (MVT) $\forall x_0 \in [\bar{x} - \varepsilon, \bar{x} + \varepsilon]$, By the Mean Value Thm (MVT) $\forall x_0 \in [\bar{x}-\varepsilon, \bar{x}+\varepsilon]$,

$$\underbrace{|\bar{x} - f(x_0)|}_{x_1} = |f(\bar{x}) - f(x_0)| = |f'(\xi_1)|\,|\bar{x} - x_0| \le c\,|\bar{x} - x_0|.$$

MVT, ξ_1 between $\bar{x}$ and x_0 so $\xi_1 \in [\bar{x}-\varepsilon, \bar{x}+\varepsilon]$

Suppose $|\bar{x} - f(x_{t-1})| \le c\,|\bar{x} - x_{t-1}|$ and $x_{t-1} \in [\bar{x}-\varepsilon, \bar{x}+\varepsilon]$.

First, $x_t = f(x_{t-1}) \in [\bar{x}-\varepsilon, \bar{x}+\varepsilon]$ because $c<1$.

Then $|\bar{x} - f(x_t)| = |f(\bar{x}) - f(x_t)| = |f'(\xi_{t+1})|\,|\bar{x} - x_t| \le c\,|\bar{x} - x_t|$.

between $\bar{x}$ and x_t

Then by induction, $|\bar{x} - f(x_t)| \le c^t |\bar{x} - x_0|$

$\Rightarrow \lim_{t\to\infty} x_t = \bar{x}$, so $\bar{x}$ is locally asymptotically stable.

Case 2: $|f'(\bar{x})| > 1$.

Then $\exists\, \varepsilon > 0$ s.t. for $x \in [\bar{x}-\varepsilon, \bar{x}+\varepsilon] \subset I$, $|f'(x)| > c > 1$.

By the MVT,

between $\bar{x}$ and x_0

$$|\bar{x} - f(x_0)| = |f'(\xi_1)|\,|\bar{x} - x_0| \ge c\,|\bar{x} - x_0|.$$

But this time, if we try to use induction, eventually $c^t|\bar{x} - x_0| > \varepsilon$.

Hence, $\exists\, t$ s.t. $|\bar{x} - f^t(x_0)| > \varepsilon$, so $\bar{x}$ is unstable. $\square$

Note: Thm 2.1 only applies if $|f'(\bar{x})| \ne 1$.

Def. 2.4 An equilibrium $\bar{x}$ of $x_{t+1} = f(x_t)$ is **hyperbolic** if

$|f'(\bar{x})| \ne 1$ and **nonhyperbolic** otherwise.

Aside: We can also generalize notions of stability to periodic solutions of period m by considering the function $f^m(x)$ instead of $f(x)$ in Thm 2.1

Thm 2.2 Suppose f' is continuous on an open interval I and

Thm 2.2 Suppose f' is continuous on an open interval I and the m-cycle $\{\bar{x}_1, f(\bar{x}_1), \ldots, f^{m-1}(\bar{x}_1)\} \subset I$.

$\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m\}$

Then the m-cycle is locally asymptotically stable if ~~for~~ some k

$$\left| \frac{d}{dx} f^m(\bar{x}_k) \right| < 1$$

and unstable if for some k

$$\left| \frac{d}{dx} f^m(\bar{x}_k) \right| > 1.$$

Aside: By the chain rule

$$\begin{aligned}
\frac{d}{dx} f^m(\bar{x}_1) &= \frac{d}{dx}\left[f^{m-1} \circ f\right](\bar{x}_1) \\
&= \frac{d}{dx}\left[f^{m-1}(f(\bar{x}_1))\right] f'(\bar{x}_1) \\
&= \frac{d}{dx}\left[f^{m-1}(\bar{x}_2)\right] f'(\bar{x}_1) \\
&= \frac{d}{dx}\left[f^{m-2}(\bar{x}_3)\right] f'(\bar{x}_1) f'(\bar{x}_2) \\
&\;\;\vdots \\
&= f'(\bar{x}_1) f'(\bar{x}_2) \cdots f'(\bar{x}_m) = \frac{d}{dx} f^m(\bar{x}_k) \quad \text{for any } k.
\end{aligned}$$

Corollary 2.1 Suppose $\{\bar{x}_1, \ldots, \bar{x}_m\}$ is an m-cycle of the difference eq $x_{t+1} = f(x_t)$. Then the m-cycle is asymptotically stable if $|f'(\bar{x}_1) \cdots f'(\bar{x}_m)| < 1$.

Ex. 2.3 $x_{t+1} = \dfrac{a x_t}{b + x_t} = f(x_t)$, $a, b > 0$. $\left(f(x) = \dfrac{ax}{b+x} \right)$

Recall that the equilibria are $\bar{x} = 0$ and $\bar{x} = a - b$.

$$f'(x) = \frac{(b+x)a - ax}{(b+x)^2} = \frac{ba}{(b+x)^2}$$

$f'(0) = \frac{a}{b}$. If $a < b$, then 0 is locally asymptotically stable
If $a > b$, then 0 is unstable.

$f'(a-b) = \frac{b}{a}$. If $a < b$, then 0 is unstable
If $a > b$, then 0 is locally asymptotically stable.

Next time: What about the nonhyperbolic case? We can't ignore higher-order terms

Ex. 2.4 Let $x_{t+1} = r - x_t^2 = f(x_t)$, $r > 0$.

Solve for $x = r - x^2$ to get equilibria.

$$x^2 + x - r = 0$$

$$\bar{x}_\pm = \frac{-1 \pm \sqrt{1+4r}}{2}$$

$f(x) = r - x^2$, so $f'(x) = -2x$.

Thus, $f'(\bar{x}_-) = 1 + \sqrt{1+4r} > 1$, so $\bar{x}_-$ is unstable.

$f'(\bar{x}_+) = 1 - \sqrt{1+4r}$

If $r < \frac{3}{4}$, $f'(\bar{x}_+) < 1$, so $\bar{x}_+$ is locally asymptotically stable.

If $r > \frac{3}{4}$, $f'(\bar{x}_+) > 1$, so $\bar{x}_+$ is unstable.

This equation also has 2-cycles.

Solve for $f^2(x) = x$

$$\Rightarrow \quad x = r - (r - x^2)^2$$

$$\Rightarrow \quad 0 = r - r^2 + 2rx^2 - x^4 - x$$

$$\Rightarrow \quad 0 = -\underbrace{(x^2 + x - r)}_{\text{equilibria}}(x^2 - x + 1 - r)$$

$$\bar{x}_{1,2} = \frac{1 \pm \sqrt{4r-3}}{2} \Bigg\} \text{ 2-cycle}$$

Let's assume $r > \frac{3}{4}$, so $\bar{x}_1, \bar{x}_2 \in \mathbb{R}$.

Exercise 2.6 Verify that the 2-cycle $\{\bar{x}_1, \bar{x}_2\}$ is locally asymptotically stable if $\frac{3}{4} < r < \frac{5}{4}$ and unstable if $r > \frac{5}{4}$.